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ON THE CLOSURE OF TRANSLATIONS IN $L^p(R_k)$

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0. Introduction

We shall discuss some aspect of "closure of translations theorem" (Cf. References listed at the end of this paper.)

Throughout this paper, we denote a conjugate exponent of p ($1 < p < \infty$) by q , that is, $1 < p < \infty$, $1 < q < \infty$, $1/p + 1/q = 1$. Let us define a sub-class W_k^q of $L^q(R_k)$ by

$$W_k^q = \left\{ \varphi(x) \in L^q(R_k) \cap L^\infty(R_k) ; \|\varphi\|_W = \sup_{\xi} \left\{ \int_{R_k} (1+|x|^\beta)^q |\varphi(x-\xi)|^q dx \right\}^{1/q} \right\}$$

where $\beta = (k-1)/p$. Note that if $k \geq 2$ and $1 < p < 2$ then

$W_k^q \subseteq L^2(R_k)$. For $\alpha > 0$, let us denote $\varphi(x)\exp(-\alpha|x|)$ by

$\varphi_\alpha(x)$ and its Fourier transform (in $L^1(R_k)$) by $\hat{\varphi}_\alpha(t)$.

We denote the zeros of the Fourier transform $\hat{f}(t)$ of $f(x) \in L^1(R_k)$ by $Z(\hat{f})$.

§1. Key Theorems.

Theorem 1. Suppose $f(x) \in L^p(R_k) \cap L^1(R_k)$, $\varphi(x) \in W_k^q$, and $f * \varphi = 0$. Then we have $\lim_{\alpha \rightarrow 0+} \hat{\varphi}_\alpha(t) = 0$ on the complement of $Z(\hat{f})$. Especially, the above limit exists uniformly on any closed interval contained in the complement of $Z(\hat{f})$.

The theorem for the case $k=1$ was proved by H. Pollard [1], and there is no essential difference between the proof for the case $k=1$ and for the case $k \geq 2$. We shall repeat the argument

due to Pollard for the sake of completeness.

Let us put

$$U(\sigma, t, y) = \int_{R_k} \varphi(x) \exp(-\sigma|x+y|) \exp(it \cdot x) dx.$$

Take any closed interval I contained in the complement of $Z(\hat{f})$.

Then we have a real number $a > 0$ such that

$$\inf_{t \in I} |\hat{f}(t)| > 2 \int_{|y| \geq a} |f(y)| dy.$$

The assumption $f * \varphi = 0$ implies

$$(1.1) \quad \int_{R_k} f(y - \xi) \exp(it \cdot (y - \xi)) U(\sigma, t, y) dy = 0.$$

Make the difference between $U(\sigma, t, \xi) \hat{f}(t)$ and (1.1).

Then we have

$$\begin{aligned} & U(\sigma, t, \xi) \hat{f}(t) - 0 \\ &= \int_{R_k} f(y - \xi) \exp(it \cdot (y - \xi)) [U(\sigma, t, \xi) - U(\sigma, t, y)] dy \\ &= \int_{|y - \xi| \leq a} + \int_{|y - \xi| > a} = J_1 + J_2, \text{ say.} \end{aligned}$$

An elementary calculation shows that

$$\begin{aligned} & |U(\sigma, t, \xi) - U(\sigma, t, y)| \\ & \leq \{M \|\varphi\|_W \sigma^{1-1/p} |y - \xi|\} \\ (1.2) \quad & = M' \sigma^{1-1/p} |y - \xi|, \end{aligned}$$

where we have denoted constants by M 's. By (1.2), we easily see that

$$|J_1| \leq M' \sigma^{1-1/p} \int_{|y| \leq a} |y| |f(y)| dy$$

and we have

$$|J_2| \leq 2 \sup_{\xi} |U(\sigma, t, \xi)| \int_{|y| > a} |f(y)| dy,$$

where we have fixed $b > 0$. Now we are ready to conclude the following inequality;

$$\begin{aligned} \sup_{\xi} |U(\sigma, t, \xi)| \left\{ |\hat{f}(t)| - 2 \int_{|y| > a} |f(y)| dy \right\} \\ \leq M'' \sigma^{1-1/p}, \end{aligned}$$

which shows the conclusion of Theorem 1.

Theorem 2. Suppose $f(x) \in L^p(R_k) \cap L^1(R_k)$, $\varphi(x) \in L^q(R_k) \cap L^2(R_k)$ and $\lim_{\sigma \rightarrow 0+} \hat{\varphi}_\sigma(t) = 0$ on the complement of $Z(\hat{f})$. Then we have $f * \varphi = 0$.

For the proof of Theorem 2, we use a technique developed by A. Beurling [2]. Put $\psi = f * \varphi$, then $\psi(x) \in L^2(R_k)$.

By the Parseval relation and the Schwarz inequality, we have

$$\begin{aligned} I(\sigma) &= \int_{R_k} |\hat{f}(t) \hat{\varphi}_\sigma(t) - \hat{\psi}_\sigma(t)|^2 dt \\ &= M \int_{R_k} dx \left| \int_{R_k} f(y) \varphi(x-y) \{ \exp(-\sigma|x-y|) - \exp(-\sigma|x|) \} dy \right|^2 \\ &= M \|f\|_1 \int_{R_k} |f(y)| dy \int_{R_k} |\varphi(x-y)|^2 \exp(-\sigma|x-y|) \\ &\quad - \exp(-\sigma|x|) \Big|^2 dx. \end{aligned}$$

Hence, by the Lebesgue theorem, $\lim_{\sigma \rightarrow 0+} I(\sigma) = 0$.

Since $\lim_{\sigma \rightarrow 0+} \|\psi - \psi_\sigma\|_2 = 0$, we get $\lim_{\sigma \rightarrow 0+} \|\hat{\psi} - \hat{f} \hat{\varphi}_\sigma\|_2 = 0$, that is,

$$\hat{\psi}(t) = \hat{f}(t) \lim_{\epsilon \rightarrow 0+} \hat{\varphi}_{\epsilon}(t) = 0. \text{ So we have } \psi = f * \varphi = 0.$$

Theorem 3. We suppose $\varphi(x) \in W_K^q \cap L^2(R_K)$ or $\varphi(x) \in L^q(R_1) \cap L^\infty(R_1)$. Let F be a closed sub-set of R_K . If $\lim_{\epsilon \rightarrow 0+} \hat{\varphi}_{\epsilon}(t) = 0$ on the complement of F , then the above limit exists uniformly on any closed interval contained in the complement of F .

Under the assumption " $\varphi \in L^q(R_1) \cap L^\infty(R_1)$ ", Theorem 3 was proved in [3]. We give a proof under the assumption " $\varphi \in W_K^q \cap L^2(R_K)$, ($k \geq 1$)". Take any closed interval I which is contained in the complement of F . Find a function $f(x) \in L^p(R_K) \cap L^1(R_K)$ such that $I \subset \text{supp}(\hat{f}) \subset CF$. By the assumption, we have $\lim_{\epsilon \rightarrow 0+} \hat{\varphi}_{\epsilon}(t) = 0$ on the complement of $Z(\hat{f})$. Since $\varphi(x) \in L^2(R_K)$, applying Theorem 2, we have $f * \varphi = 0$. Now we can appeal to Theorem 1, and we have $\lim_{\epsilon \rightarrow 0+} \hat{\varphi}_{\epsilon}(t) = 0$ uniformly on $t \in I$.

§ 2. Closure problems (1).

We shall introduce several notions for the discussion of "closure of translations problem".

A linear subfamily W of L^q may introduce such the weakest topology into $L^p(R_K)$ that it makes only elements of W continuous linear functionals on $L^p(R_K)$. We call such a topology mentioned above by " W -topology". We denote the closure of the linear manifold spanned by the translates of $f(x) \in L^p(R_K)$ by $T[f; W]$, where the closure is considered under W -topology.

A closed sub-set F of R_k is said to be a $(U;W)$ -set if the relations

$$\lim_{\sigma \rightarrow 0+} \hat{\varphi}_\sigma(t) = 0 \text{ on } t \in CF, \quad \varphi \in W \subset L^q(R_k)$$

imply that $\varphi(x) = 0$, a.e..

Using the notions introduced above, we can interpret Theorems 1 and 2 in the following forms.

Theorem 4. Let $f(x) \in L^p(R_k) \cap L^1(R_k)$. If $Z(\hat{f})$ is a $(U;W_k^q)$ -set, then $T[f; W_k^q] = L^p(R_k)$.

Theorem 5. Let $f(x) \in L^p(R_k) \cap L^1(R_k)$. If $T[f; L^q(R_k) \cap L^2(R_k)] = L^p(R_k)$, then $Z(\hat{f})$ is a $(U; L^q(R_k) \cap L^2(R_k))$ -set.

Theorem 6. Let $f(x) \in L^p(R_k) \cap L^1(R_k)$. Then

$$T[f; W_k^q \cap L^2(R_k)] = L^p(R_k)$$

if and only if $Z(\hat{f})$ is a $(U; W_k^q \cap L^2(R_k))$ -set.

§3. Closure problems (2).

According to R.E. Edwards [4], we shall introduce a notion of thin-set. A closed sub-set F of R_k is said to be $(p;W)$ -thin if the relations

$$\text{supp}(\hat{\varphi}) \subset F, \quad \varphi \in W \subset L^q(R_k) \cap L^\infty(R_k)$$

imply that $\varphi(x) = 0$, a.e., where $\text{supp}(\hat{\varphi})$ means the

support of the generalized Fourier transform (i.e. a pseudo measure) $\hat{\varphi}$ of $\varphi \in L^\infty(R_k)$.

Theorem 7. A closed sub-set $F \subseteq R_k$ is $(p; W_k^q \wedge L^2(R_k))$ -thin, if and only if F is a $(U; W_k^q \wedge L^2(R_k))$ -set.

For the case $k=1$, we can exclude the word " $L^2(R_k)$ " from the above statement (cf. Edwards [4]), that is, the notion $(U; L^q(R_1) \wedge L^\infty(R_1))$ is equivalent to the notion $(p; L^q(R_1) \wedge L^\infty(R_1))$.

For the proof of Theorem 7, suppose F is $(p; W_k^q \wedge L^2(R_k))$. In order to show that F is $(U; W_k^q \wedge L^2(R_k))$, it is enough to show that the relations

$$\lim_{\epsilon \rightarrow 0+} \hat{\varphi}_\epsilon(t) = 0 \text{ on } t \in CF, \varphi \in W_k^q \wedge L^2(R_k)$$

imply $\text{supp}(\hat{\varphi}) \subseteq F$. For this purpos, take any closed interval $I \subset CF$. Consider any function $\psi \in \mathcal{S}$ (the Schwartz space) such that $\text{supp}(\psi) \subseteq I$. Since $\lim_{\epsilon \rightarrow 0+} \hat{\varphi}_\epsilon = \hat{\varphi}$ in \mathcal{S}' (the temperate distribution space, the dual space of \mathcal{S}), we have

$$\langle \hat{\varphi}, \psi \rangle = \lim_{\epsilon \rightarrow 0+} \int_{R_k} \hat{\varphi}_\epsilon(t) \psi(t) dt = \lim_{\epsilon \rightarrow 0+} \int_I \hat{\varphi}_\epsilon(t) \psi(t) dt .$$

By Theorem 3, $\lim_{\epsilon \rightarrow 0+} \hat{\varphi}_\epsilon(t) = 0$, uniformly on $t \in I$. Therefore, we have $\langle \hat{\varphi}, \psi \rangle = 0$, which shows that $\text{supp}(\hat{\varphi}) \subseteq F$.

Conversely, we suppose that F is $(U; W_k^q \wedge L^2(R_k))$.

In order to show that F is $(p; W_k^q \wedge L^2(R_k))$, we have

to show that the relations

$$\text{supp}(\hat{\varphi}) \subseteq F, \quad \varphi \in W_k^q \wedge L^2(R_k)$$

imply $\lim_{\delta \rightarrow 0+} \hat{\phi}_{\delta}(t) = 0$ on $t \in CF$.

Let $Sp(\varphi)$ be the spectrum of $\varphi(x) \in L^{\infty}(R_k)$, that is,

$$Sp(\varphi) = \bigcap_{g \in J} Z(\hat{g}), \quad J = \{g \in L^1(R_k) ; g * \varphi = 0\}.$$

Then $Sp(\varphi) = \text{Supp}(\hat{\varphi})$ (cf. J.P.Kahane [5] and Edwards[4]).*)

Since $\text{supp}(\hat{\varphi}) \subseteq F$, we have $CF \subseteq C Sp(\varphi)$. Take any point

$t_0 \in CF \subseteq C Sp(\varphi)$, then, by the definition of $Sp(\varphi)$, there

exists $f(x) \in L^1(R_k)$ such that $f * \varphi = 0$ but $\hat{f}(t_0) \neq 0$.

We may suppose that $f(x) \in L^p(R_k) \cap L^1(R_k)$. Now apply Theorem 1,

then we have $\lim_{\delta \rightarrow 0+} \hat{\phi}_{\delta}(t_0) = 0$, which completes the proof.

In the second part of the above argument, we did not use L^2 -property of φ . In fact, we have proved

Theorem 8. If F is $(U; W_k^q)$, then F is $(p; W_k^q)$.

§4. Uniqueness theorem for the Poisson summability of trigonometric integrals.

We may interpret Theorem 7 in the following form:

Theorem 9. Let $\varphi(x) \in W_k^q \cap L^2(R_k)$ and F be $(p; W_k^q \cap L^2(R_k))$.

If $\lim_{\delta \rightarrow 0+} \int_{R_k} \varphi(x) \exp(it \cdot x) \exp(-\delta|x|) dx = 0$

on $t \in CF$, then $\varphi(x) = 0$, a.e. . (For the case $k=1$, we can exclude the word " $L^2(R_k)$ " from the above assumptions.)

R.E.Edwards [4] gave several examples of p -thin sets.

For example, any discrete set is $(p; L^q(R_k) \cap C_0(R_k))$ -thin,

*) C.S.Herz [7] reduced the closure problem to the spectral synthesis problem.

where $C_0(R_k)$ denotes the space of continuous functions on R_k which tends to zero at infinity. Combined this fact and Theorem 9, we have the following uniqueness theorem:

Theorem 10. A discrete set is a set of uniqueness for the Poisson summability of trigonometric integrals of $\varphi \in W_k^q \cap C_0(R_k) \cap L^2(R_k)$ or $\varphi \in L^q(R_1) \cap C_0(R_1)$.

From the fact that any discrete set is $(p; L^q(R_k) \cap C_0(R_k))$ -thin, Edwards concluded that if $f \in L^p(R_k) \cap L^1(R_k)$ and if $Z(\hat{f})$ is discrete then $T[f; L^q] = L^p(R_k)$. This was proved also by I.E. Segal [6].

§5. Simple proof of uniqueness theorem for the Poisson summability of trigonometric integrals.

The proof of Theorem 7 suggests us a simple proof of uniqueness theorem for the Poisson summability of trigonometric integrals: The following simple result is a key for the problem.

Theorem 11. Suppose $\varphi \in L^\infty(R_k)$. If $\lim_{\epsilon \rightarrow 0+} \hat{\varphi}_\epsilon(t) = 0$ for all $t \in R_k$, and if the above limit exists uniformly on any finite closed interval in R_k , then $\varphi = 0$, a.e. .

The proof of Theorem 11 is nothing but the repeat of the proof of Theorem 7. In fact, we have $\lim_{\epsilon \rightarrow 0+} \hat{\varphi}_\epsilon = \hat{\varphi}$, distributionally. The assumption of uniformity implies that $\lim_{\epsilon \rightarrow 0+} \hat{\varphi}_\epsilon = 0$. This means that $\hat{\varphi} = 0$, that is, $\varphi = 0$, a.e. .

As a consequence of Theorem 11, when we want to conclude " $\varphi = 0$, a.e. " from " $\lim_{\epsilon \rightarrow 0+} \hat{\varphi}_\epsilon(t) = 0$, everywhere " , it is enough to show that the above limit exists uniformly on any

closed interval in R_k . Therefore, combine Theorem 3 and Theorem 11, we have

Theorem 12. Suppose $\varphi \in W_k^q \cap L^2(R_k)$, $k \geq 1$. If $\lim_{\epsilon \rightarrow 0+} \hat{\varphi}_\epsilon(t) = 0$, everywhere on R_k , then $\varphi = 0$, a.e. .

We have to remark that we do not drop the assumption " L^2 " from the above theorem even for the case $k = 1$. The situation is as follows: For the proof of Theorem 3 under the assumption " $\varphi \in L^q(R_1) \cap L^\infty(R_1)$ ", we need the uniqueness theorem. (Cf. Proof of Theorem B in [1] and Proof of Lemma 2 in [3]). However, when we prove Theorem 3 under the assumption " $\varphi \in W_k^q \cap L^2(R_k)$, $k \geq 1$ ", we do not need the uniqueness theorem.

When the case $k = 1$, we can generalize Theorem 12 in the following way:

Theorem 13. Suppose $\varphi \in L^q(R_1) \cap L^\infty(R_1)$, and

$$(5.1) \quad \int_I |\hat{\varphi}_\epsilon(t)|^2 dt \leq C(I) < \infty, \text{ for } \epsilon > 0 \text{ and for any}$$

finite interval I in R_1 ,

where $C(I)$ is constant depending only on I .

If $\lim_{\epsilon \rightarrow 0+} \hat{\varphi}_\epsilon(t) = 0$, everywhere, then $\varphi = 0$, a.e. .

For the proof of Theorem 13, we have to show that the limit $\lim_{\epsilon \rightarrow 0+} \hat{\varphi}_\epsilon(t) = 0$ exists uniformly on I . In order to establish the above, we need a theorem corresponding to Theorem 3, and hence we want to have a result of type of Theorem 2. For this purpose, just repeat the argument in [2], then we have

Theorem 14. Suppose $|x|^{1/2}f(x) \in L^1(R_1)$, $\varphi \in L^\infty(R_1)$ and (5.1). If $\lim_{\epsilon \rightarrow 0+} \hat{\varphi}_\epsilon(t) = 0$ on the complement of $Z(\hat{f})$, then $f * \varphi = 0$.

From Theorem 14 and Theorem 1 of the case $k = 1$, we have the following:

Theorem 15. Suppose $\varphi \in L^q(R_1) \cap L^\infty(R_1)$ and (5.1). Let F be a closed subset of R_1 . If $\lim_{\epsilon \rightarrow 0+} \hat{\varphi}_\epsilon(t) = 0$ on the complement of F , then the above limit exists uniformly on any closed interval contained in the complement of F .

From the above setting, we can conclude Theorem 13.

Remark that (5.1) holds if $\varphi \in L^2(R_1)$. This follows from the Parseval relation.

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